On K-node Survivable Power Efficient Topologies in Wireless Networks with Sectored Antennas

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Abstract

We consider the problem of survivable minimum power bidirectional topology optimization in wireless networks with sectored antennas. In this paper, we take an algebraic view of graph connectivity, which is defined as the second smallest eigenvalue of the laplacian matrix of a graph. We propose a (sub-optimal) centralized heuristic procedure for constructing power efficient K-node connected topologies. The procedure comprises a construction phase and an improvement phase. The construction phase is based on Kruskal's algorithm for the minimum spanning tree (MST) problem. However, unlike Kruskal's MST algorithm which chooses minimum cost edges from a set of edge weights, our algorithm uses an incremental cost mechanism to select edges. This incremental cost mechanism is motivated by the inherently broadcast nature of the wireless medium. The topology improvement phase is used to remove non-essential edges from the construction phase, without affecting the desired connectivity.

1 Introduction

We consider the problem of survivable power efficient bidirectional topology in multihop wireless networks where individual nodes are typically equipped with limited capacity batteries and therefore have a restricted communication radius. Topology control is one of the most fundamental and critical issues in multihop wireless networks which directly affect the network performance. In wireless networks, topology control essentially involves choosing the right set of transmitter powers to maintain adequate network connectivity. Incorrectly designed topologies can lead to higher end-to-end delays and reduced throughput in error-prone channels. In energy-constrained networks where replacement or periodic maintenance of node batteries is not feasible, the issue is all the more critical since it directly impacts the network lifetime.

In a seminal paper on topology control using transmit power control in wireless networks, Ramanathan and Rosales-Hain [1] approached the problem from an optimization viewpoint and showed that a network topology which minimizes the maximum transmitter power allocated to any node can be constructed in polynomial time. This is a critical criterion in battlefield applications since using higher transmitter power increases the probability of detection by enemy radar. They also proposed optimal polynomial time algorithms for both 1-connected and 2-connected topologies. In this paper, we consider a different version of the power efficient topology optimization problem, that of minimizing the total transmit power, as opposed to minimizing the maximum transmitter power. Minimizing the total transmit power has the effect of limiting the total interference power in the network. It has been shown in [2] that this problem is NP-complete for the special case of 1-connected topologies with omnidirectional (or, single sector) antennas. Related work in the area of 1-connected minimum power topology optimization include [3], [4] and [5], all of which propose distributed algorithms. Specifically, [3] proposes a cone-based distributed algorithm which relies only on angle-of-arrival estimates to establish a power efficient connected topology. Huang *et al* [4] describe a distributed protocol which is designed for sectored antenna systems. The work in [5] explores the use of relative neighborhood graphs (RNG) for topology control and suggests an algorithm for distributed computation of the RNG.

An excellent survey of the issues and challenges related to survivability of wireless networks can be found in [6]. One of the key research issues identified in [6] is "establishing and maintaining survivable topologies that strive to keep the network connected even under attack". The paper also argues for the need for research on power efficient topologies while adhering to certain connectivity constraints.

In this paper, we describe a sub-optimal centralized heuristic for constructing and improving power efficient K-node connected topologies. The construction phase builds an initial K-connected topology and is based on Kruskal's algorithm for the minimum spanning tree problem. However, unlike Kruskal's algorithm which chooses minimum cost edges from a set of edge weights, our algorithm uses an incremental cost mechanism to select edges. This incremental cost mechanism is motivated by the inherently broadcast nature of the wireless medium. The topology improvement phase is used to remove non-essential edges from the construction phase, without affecting the desired connectivity. As opposed to standard graph theoretic tests of K-connectivity, in this paper we adopt an algebraic notion of K-connectivity which is defined as the second smallest eigenvalue of the laplacian matrix of a graph. Though not the subject of this paper, such an approach allows for solving a relaxation of the power efficient K-connected topological optimization problem efficiently within a semi-definite programming (SDP) framework. This will be discussed in a subsequent paper.

The rest of the paper is organized as follows. In Section 2, we describe the network model and outline our assumptions. In Section 3, we summarize some definition and theorems related to algebraic graph theory and in Section 4 we formally define the problem. Sections 5 and 6 describe the topology construction and improvement algorithms. The algorithms are illustrated with an example in Section 7 and detailed simulation results are presented in Section 8.

2 Network Model and Assumptions

In this section, we outline our network model and assumptions. At the outset, though, we would like to mention that the terms *K-node (edge) survivability* and *K-node (edge) connectivity* have been used interchangeably throughout this paper. Similarly, the term *symmetric (or bidirectional) topology* has been used interchangeably with *undirected (or bidirected) graph* in a graph theoretic context. Also, we will consistently use the notations $\vec{x}$ and $\mathbf{x}$ to denote a vector and a matrix respectively.

1. We consider a fixed N-node wireless network in a two-dimensional plane.

2. All nodes are equipped with limited capacity batteries.

3. All nodes are assumed to have identical S-sector antennas. The number of sectors, S, is related to the beamwidth, θ (in degrees), as follows:

$$S = 360/\theta \tag{1}$$

   Each sector is assumed to span the angular region $[(s-1)360/\theta, (s)360/\theta]$ in the 2-D plane, where $1 \leq s \leq S$ is the sector number. Note that $\theta = 360$ ($\Rightarrow S = 1$) corresponds to an omnidirectional antenna.

4. We ignore sidelobe effects and assume that when sector s is switched on, 100% of the radiated power is confined within that sector, providing an uniform gain within the angular region spanned by the sector.

5. The efficiency of the antennas is assumed to be 100%; $i.e.$, all the input power is assumed to be converted to radiated power.

6. We assume that both transmission and reception is directional.

7. Following our above assumptions, the transmitter power at i necessary to support the link $i \rightarrow j$[1], $\mathbf{P}_{ij}$, is proportional (accounting for link/antenna gains and other factors) to d_{ij}^{α}/S^2, where d_{ij} is the Euclidean distance

[1] In this paper, the notation $i \leftrightarrow j$ is used to denote a bidirectional link between nodes i and j while a directed link from i to j is represented by $i \rightarrow j$. The notation (i, j) is used to refer to the node pair.

between nodes i and j and α is the channel loss exponent, typically between 2 and 4. Without any loss of generality, we set the proportionality constant equal to 1 and therefore:

$$\mathbf{P}_{ij} = d_{ij}^{\alpha}/S^2 \tag{2}$$

For energy constrained networks, it is desirable that the network topology be optimized taking into account battery residual capacities. This can be accomplished by redefining $\mathbf{P}_{ij}$ as follows:

$$\mathbf{P}_{ij} = C_i^{-\beta}(t)\; C_j^{-\beta}(t)\; (d_{ij}^{\alpha}/S^2) \tag{3}$$

where $C_i(t)$ is the normalized battery residual capacity of node i at time t ($0 \leq C_i(t) \leq 1$) and β is a scaling factor, $\beta \geq 1$. Although we have assumed identical scaling factors for both the transmitter and the receiver, this assumption can be relaxed to account for different scaling factors.

8. We assume that $\mathbf{P}_{ij} = \mathbf{P}_{ji}, \forall (i,j) \in \mathcal{N}$ where $\mathcal{N}$ is the set of all nodes in the network and $N = |\mathcal{N}|$.

9. There is a constraint on the maximum power level per sector which a node can use for transmission and that this parameter is identical for all nodes. We denote this maximum power level by P^{max}.

   If $Y_{i,s}$ is the transmission power cost corresponding to sector s of node i, we have:

$$0 \leq Y_{i,s} \leq P^{max} : \forall i \in \mathcal{N}, 1 \leq s \leq S \tag{4}$$

10. Let $\mathcal{E}$ be the set of all bidirected edges and $E = |\mathcal{E}|$. Using the transmitter power constraint, the set $\mathcal{E}$ is given by:

$$\mathcal{E} = \{(i \leftrightarrow j) :\; (i,j) \in \mathcal{N},\; i \neq j,\; P^{max} \geq \mathbf{P}_{ij}, \mathbf{P}_{ji}\} \tag{5}$$

   The third condition on the right hand side of (5) enforces bidirectionality of edges based on the maximum sector power constraint. For the sake of notational simplicity, we will also use the set $\mathcal{E}$ to refer to all directed edges, $\{i \to j\}$, in the graph, since:

$$i \leftrightarrow j \in \mathcal{E} \Longleftrightarrow i \to j \in \mathcal{E} \text{ and } j \to i \in \mathcal{E} \tag{6}$$

   We will refer to the graph $G = (\mathcal{N}, \mathcal{E})$ as the *reachability graph* of the network.

11. We assume that the parameter P^{max} is such that no node can reach all other nodes in the network. That is, for each i, there is at least one j such that $i \to j \notin \mathcal{E}$.

12. The *link cost matrix* is an $N \times N$ symmetric matrix, $\mathbf{C}$, whose (i,j)-th element is given by

$$\mathbf{C}_{ij} = \begin{cases} \mathbf{P}_{ij}, & \text{if } i \to j \in \mathcal{E} \\ \infty, & \text{otherwise} \end{cases} \tag{7}$$

   where $\mathbf{P}_{ij}$ is as defined in (2) or (3).

3 Some Definitions and Theorems

We now provide a compilation of essential definitions and theorems which we will use in developing a mathematical formulation of the survivable power efficient topological optimization problem. The theorems and, in most cases, their proofs can be found in the references cited.

Definition 1 *An undirected graph, $G = (\mathcal{N}, \mathcal{E})$, where $\mathcal{N}$ is the set of nodes and $\mathcal{E}$ is the set of edges, is simple if (a) there is at most one edge between any pair of nodes and (b) there are no loops; $i.e.$, there are no edges of the form (i,i).*

Definition 2 *The complement of a simple graph G is the graph, $\bar{G}$, with the same node set as G, but whose edge set is the complement of the edge set of G; $i.e.$, $\bar{G} = (\mathcal{N}, \bar{\mathcal{E}})$. The complement of the edge set of G is the set of all possible edges on the vertex set of G which are not in $\mathcal{E}$.*

Definition 3 *An undirected graph is connected if there is a path between any pair of nodes i and j in $\mathcal{N}$.*

Definition 4 *The node connectivity of a graph G, denoted by $\kappa_n(G)$, is equal to the minimum number of nodes whose deletion from G causes the graph to be disconnected or reduces it to a 1-node graph.*

Definition 5 *A graph G is K-node connected if $\kappa_n(G) \geq K$.*

Definition 6 *The edge connectivity of a graph G, denoted by $\kappa_e(G)$, is equal to the minimum number of edges whose deletion from G causes the graph to be disconnected.*

Definition 7 *A graph G is K-edge connected if $\kappa_e(G) \geq K$.*

Definition 8 *The union of two graphs $G_1 = (\mathcal{N}_1, \mathcal{E}_1)$ and $G_2 = (\mathcal{N}_2, \mathcal{E}_2)$, denoted by $G_1 \cup G_2$, is the graph defined by the node set $\mathcal{N}_1 \cup \mathcal{N}_2$ and the edge set $\mathcal{E}_1 \cup \mathcal{E}_2$.*

Definition 9 *The join of two graphs $G_1 = (\mathcal{N}_1, \mathcal{E}_1)$ and $G_2 = (\mathcal{N}_2, \mathcal{E}_2)$, denoted by $G_1 \vee G_2$, is the graph obtained from $G_1 \cup G_2$ by adding new edges from each node in G_1 to every node in G_2.*

Definition 10 *The adjacency matrix of a simple graph G, denoted by $\mathbf{A}(G)$, is a $N \times N$ ($N = |\mathcal{N}|$) binary matrix such that:*

$$\mathbf{A}_{ij}(G) = \begin{cases} 1, & \textit{if } (i \leftrightarrow j) \in \mathcal{E} \\ 0, & \textit{otherwise} \end{cases} \tag{8}$$

The adjacency matrix of simple undirected graphs is symmetric and has all diagonal elements equal to 0. Additionally, if the graph is connected and $N \geq 2$, its adjacency matrix is irreducible; $i.e.$, there exists a permutation matrix $\mathbf{P}$ such that:

$$\mathbf{P}'\mathbf{A}(G)\mathbf{P} = \begin{bmatrix} \mathbf{A}_{11}(G) & \mathbf{A}_{12}(G) \\ \mathbf{0} & \mathbf{A}_{22}(G) \end{bmatrix} \tag{9}$$

Definition 11 *The degree (or, valency) of any node i in an undirected graph G, denoted by deg_i, is equal to the number of edges incident on i; $i.e.$, $deg_i = \sum_j \mathbf{A}_{ij}(G) = \sum_j \mathbf{A}_{ji}(G)$.*

Definition 12 *The laplacian matrix of a graph G, denoted by $\mathbf{L}(G)$, is a $|\mathcal{V}| \times |\mathcal{V}|$ matrix such that:*

$$\mathbf{L}_{ij}(G) = \begin{cases} deg_i, & \textit{if } j = i \\ -\mathbf{A}_{ij}(G), & \textit{otherwise} \end{cases} \tag{10}$$

For connected simple undirected graphs, $\mathbf{L}(G)$ has the following properties:

- *It is symmetric and all its off-diagonal elements belong to the set {0, -1}.*
- *Its diagonal elements are positive.*
- *All row and column sums are equal to 0 ($\Rightarrow \mathbf{L}(G)$ is singular). Specifically, $\mathbf{L}_{ii}(G) = \sum_{j \neq i} |\mathbf{L}_{ij}(G)|, \ \forall i$.*

Definition 13 *The laplacian eigenvalues of G are the roots of the characteristic polynomial of $\mathbf{L}(G)$.*

Since $\mathbf{L}(G)$ is symmetric, all its eigenvalues are real. In this paper, we will assume throughout that the eigenspectrum of any matrix is ordered according to magnitude, from the smallest to the greatest: $|\lambda_1| \leq |\lambda_2| \leq \cdots \leq |\lambda_N|$ ($N = |\mathcal{N}|$), and repeated according to their multiplicity. The notation λ_k will be used to denote the k-th "smallest" (in terms of absolute value) eigenvalue. For the laplacian matrix, all its eigenvalues are nonnegative (see Theorem 1) and therefore $\lambda_k = |\lambda_k|; 1 \leq k \leq N$. The eigenvalue with the largest magnitude, λ_N, is known as the *spectral radius* of $\mathbf{L}(G)$.

The following theorems relate the connectivity of an undirected graph to the eigenspectrum of its laplacian matrix.

Theorem 1 *The smallest eigenvalue of the laplacian of an undirected graph G is equal to 0 ($i.e.$, $\lambda_1(\mathbf{L}(G)) = 0$) and its corresponding eigenvector is the unit vector $\vec{e}$.*

Moreover, the multiplicity of 0 as an eigenvalue of $\mathbf{L}(G)$ is equal to the number of connected components of G (see Section 2 of [11]). Consequently, if the graph G is disconnected, $\lambda_2(\mathbf{L}(G)) = 0$.

Theorem 2 *For any undirected graph $G = (\mathcal{N}, \mathcal{E})$, the second eigenvalue of its laplacian is upper bounded by its node connectivity, which in turn is upper bounded by its edge connectivity [11].*

$$\lambda_2(\mathbf{L}(G)) \leq \kappa_n(G) \leq \kappa_e(G) \tag{11}$$

Moreover, if G is not complete[2]*, $i.e.$, there is at least one pair of nodes i and j such that $(i \to j) \notin \mathcal{E}$ (see pp. 289 of [13]), its edge connectivity is upper bounded by the minimum node degree.*

$$\lambda_2(\mathbf{L}(G)) \leq \kappa_n(G) \leq \kappa_e(G) \leq \mathit{min}_{i \in \mathcal{N}} \ (deg_i(G)) \tag{12}$$

Theorem 3 *Let G be a simple connected graph of order N and $\bar{G}$ its complement. Then, the sum of the laplacians of G and $\bar{G}$ is given by [12]:*

$$\mathbf{L}(G) + \mathbf{L}(\bar{G}) = N \cdot \mathbf{I} - \mathbf{J} \tag{13}$$

where $\mathbf{I}$ is the identity matrix and $\mathbf{J}$ is a matrix all of whose entries are equal to 1. Also, the second eigenvalue of $\mathbf{L}(G)$ is related to the spectral radius of $\mathbf{L}(\bar{G})$ as follows:

$$\lambda_2(\mathbf{L}(G)) = N - \lambda_N(\mathbf{L}(\bar{G})) \tag{14}$$

Theorem 4 *Let G' be the graph obtained by adding an edge e to the undirected graph G; $i.e.$, $G' = G + e$. Then, the eigenvalues of the laplacians of G and G' interlace as follows [11]:*

$$\lambda_1(\mathbf{L}(G)) \leq \lambda_1(\mathbf{L}(G')) \leq \lambda_2(\mathbf{L}(G)) \leq \lambda_2(\mathbf{L}(G')) \leq \cdots \leq \lambda_n(\mathbf{L}(G)) \leq \lambda_n(\mathbf{L}(G')) \tag{15}$$

4 Problem Statement

Let $\vec{Y}$ be a vector of node transmission powers, the element Y_i representing the total transmission power cost of node i. For an S-sector antenna, Y_i can be written as:

$$Y_i = \sum_{s=1}^{S} Y_{i,s} \tag{16}$$

where $Y_{i,s}$ is the transmission power cost corresponding to sector s of node i.

The objective function for the K-node survivable minimum power topological (MPT) optimization problem is:

$$\text{MPT:} \ \ minimize \sum_{i=1}^{N} Y_i = minimize \sum_{i=1}^{N} \sum_{s=1}^{S} Y_{i,s} \tag{17}$$

which is to be solved subject to the constraint that the solution be a K-node connected simple graph. For the rest of this paper, we concentrate solely on the node connectivity case, although the heuristic algorithm we discuss later can be extended straightforwardly to the K-edge connected case. Henceforth, we will refer to K-node connectivity as simply K-connectivity.

Note that instead of minimizing the total transmit power, we could have minimized the *per-node* maximum transmitter power:

$$minimize \left(\max_i \left\{ \sum_{s=1}^{S} Y_{i,s} : i \in \mathcal{N} \right\} \right) \tag{18}$$

[2]This caveat holds as per assumption (11), Section 2.

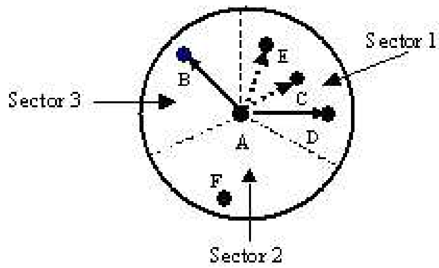

Figure 1: Illustrating link support with a 3-sector antenna. Nodes C, D and E are located in the same sector, w.r.t A, D being the farthest. Existence of the link $A \rightarrow D$ implies the existence of links $A \rightarrow C$ and $A \rightarrow E$. The total transmission cost of node A is: $Y_A = Y_{A,1} + Y_{A,3}$, where $Y_{A,1} = max(\mathbf{P}_{AC}, \mathbf{P}_{AD}, \mathbf{P}_{AE}) = \mathbf{P}_{AD}$ and $Y_{A,3} = \mathbf{P}_{AB}$. Since sector 2 is not used, $Y_{A,2} = 0$.

or the *per-sector* maximum transmitter power (MMPT):

$$\text{MMPT: } minimize\,(\max_i \{Y_{i,s} : i \in \mathcal{N}; 1 \leq s \leq S\}) \tag{19}$$

The objective functions (18) and (19) are equivalent for networks with omnidirectional antennas ($i.e.$, $S = 1$). It has been shown by Ramanathan and Rosales-Hain [1] that (18) can be solved optimally in polynomial time for an omnidirectional antenna system. The algorithm proposed by them is also applicable to the sectored antenna case, if equation (2) is used to compute edge costs. However, Clementi *et al* [2] have shown that the objective function in (17) is NP-complete for $S = 1$ and $K = 1$. Consequently, it can be inferred that the general S-sector K-node survivable optimization problem is NP-complete too.

In a wireless network with sectored antennas, the existence of a link from node i to node j also implies the existence of links from i to all nodes which are geometrically closer to i than j and are located in the same sector as j, with respect to i. We will refer to this property as the *sectored wireless advantage property*. For example, in Figure 1, nodes C, D and E are all located in the same sector w.r.t node A, node D being the farthest. Existence of the link $A \rightarrow D$ therefore implies the existence of links $A \rightarrow C$ and $A \rightarrow E$. The total transmit power cost of node A is $Y_A = Y_{A,1} + Y_{A,3}$, where $Y_{A,1} = max(\mathbf{P}_{AC}, \mathbf{P}_{AD}, \mathbf{P}_{AE}) = \mathbf{P}_{AD}$ and $Y_{A,3} = \mathbf{P}_{AB}$. Since sector 2 is not used, $Y_{A,2} = 0$.

Let $ne(i, s)$ be the set of neighbors of node i which are within radio range of i and located within the same sector, s, w.r.t node i. For example, in Figure 1, $ne(A, 1) = \{C, E, D\}$, $ne(A, 2) = \{F\}$ and $ne(A, 3) = \{B\}$. Using the sectored wireless advantage property (see Figure 1), the variable $Y_{i,s}$ can be expressed as:

$$Y_{i,s} = \max_j \{X_{ij}\mathbf{C}_{ij} : j \in ne(i, s),\ (i \rightarrow j) \in \mathcal{E}\} \tag{20}$$

where X_{ij} is a binary variable equal to 1 if the edge $i \rightarrow j$ is included in the solution and 0 otherwise. Of course, since we seek a symmetric topology, any optimization model should include the following forcing constraints:

$$X_{ij} = X_{ji} : \forall\ (i \rightarrow j) \in \mathcal{E}$$

We now consider node connectivity constraints which are of the form:

$$\kappa_n(G) = K \tag{21}$$

where K is the desired connectivity parameter. For example, $K = 2$ if a biconnected topology is desired, $K = 3$ for a triconnected topology, etc. For $K = 1$, a mixed integer linear programming model is discussed in [9] for solving the MPT problem optimally. However, modelling connectivity constraints within a linear program for $K > 1$ involves significant computational complexity. We therefore adopt a heuristic approach using the following constraint in place of (21):

$$\lambda_2(\mathbf{L}(G)) > K - 1 \tag{22}$$

where $\lambda_2(\mathbf{L}(G))$ is the second smallest eigenvalue of the laplacian matrix of G, also known as the *algebraic connectivity of* G [14]. However, to avoid numerical precision issues, it is preferable to work with the following constraint in place of (22):

$$\lambda_2(\mathbf{L}(G)) \geq K - 1 + \epsilon \tag{23}$$

where ϵ is a small positive number on the order of, say, 10^{-10}.

Note that, from Theorem 2, $\lambda_2(\mathbf{L}(G)) > K - 1 \Rightarrow \kappa_n(G) \geq \lambda_2(\mathbf{L}(G)) > K - 1 \Rightarrow \kappa_n(G)$ is at least as great as K, considering the integrality property of $\kappa_n(G)$. For 1-connectedness (or, simple connectivity), the algebraic condition that needs to be satisfied is therefore $\lambda_2(\mathbf{L}(G)) > 0$.

In order to provide a graph theoretic interpretation of the K-node survivable MPT problem, we first define $\Omega(G)$ to be the set of all simple undirected graphs defined on the node set $\mathcal{N}$ and edge set $\mathcal{E}$ (5) which are K-node connected. For any graph $G \in \Omega(G)$, $\mathbf{A}_{ij}(G) \in \{0, 1\}$, $\forall (i \rightarrow j) \in \mathcal{E}$, and equal to 0 otherwise. If $\widehat{G} \in \Omega(G)$ is the optimal graph, it can be seen from (17) and (20) that $\widehat{G}$ minimizes

$$\sum_{i=1}^{N} \sum_{s=1}^{S} \max_j \{\mathbf{C}_{ij}\mathbf{A}_{ij}(G) : j \in ne(i,s)\} \tag{24}$$

when the optimization is carried out over the set $\Omega(G)$, subject to the connectivity constraint (22).

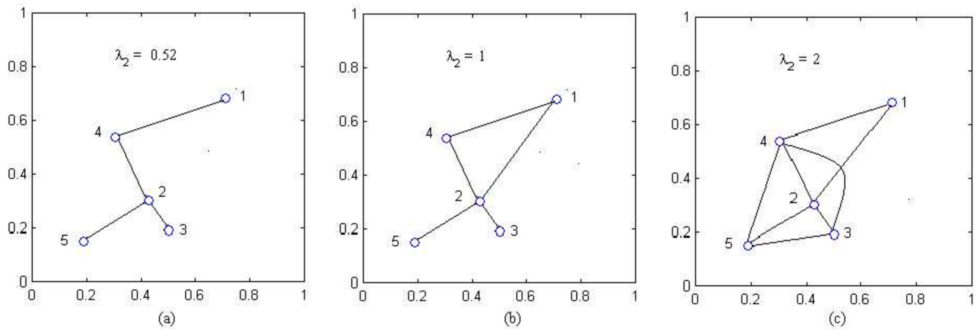

Figure 2: (a) Graph G_1 is 1-connected since removal of node 4 disconnects node 2 from 1, 3 and 5. However, $\lambda_2(G_1) = 0.52$. (b) Graph G_2, obtained by adding the edge $1 \leftrightarrow 2$ to G_1, $\lambda_2(G_2) = 1$. It can be checked that $\kappa_n(G_2) = 1$ since removal of node 2 partitions the residual graph. (c) Graph G_3, obtained by adding the edges $5 \leftrightarrow 4$, $5 \leftrightarrow 3$ and $3 \leftrightarrow 4$ to G_2. This graph is 2-connected since removal of nodes 2 and 4 partitions the residual graph into the two components $\{1\}$ and $\{3, 5\}$. It is also verified by the second eigenvalue of its laplacian, which is equal to 2.

To illustrate the relationship between graph connectivity and the second eigenvalue of its laplacian, consider the graphs G_1, G_2 and G_3 in Figures 2a, 2b and 2c. The graph G_2 is obtained by adding the edge $1 \leftrightarrow 2$ to G_1 and G_3 is obtained by adding edges $5 \leftrightarrow 4$, $5 \leftrightarrow 3$ and $3 \leftrightarrow 4$ to G_2. The laplacians of these graphs are shown in (25), (26) and (27).

$$\mathbf{L}(G_1) = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 3 & -1 & -1 & -1 \\ 0 & -1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{bmatrix} \tag{25}$$

$$\mathbf{L}(G_2) = \begin{bmatrix} 2 & -1 & 0 & -1 & 0 \\ -1 & 4 & -1 & -1 & -1 \\ 0 & -1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{bmatrix} \tag{26}$$

$$\mathbf{L}(G_3) \quad = \begin{bmatrix} 2 & -1 & 0 & -1 & 0 \\ -1 & 4 & -1 & -1 & -1 \\ 0 & -1 & 3 & -1 & -1 \\ -1 & -1 & -1 & 4 & -1 \\ 0 & -1 & -1 & -1 & 3 \end{bmatrix} \tag{27}$$

On computing the eigenvalues of (25), it can be seen that $\lambda_2(\mathbf{L}(G_1)) = 0.52$ even though the graph G_1 is 1-connected, $i.e.$, $\kappa_n(G_1) = 1$. However, if the edge $1 \leftrightarrow 2$ is added to G_1, the second eigenvalue of the laplacian of the resulting graph (Figure 2b) increases to 1, equal to the node connectivity of G_2. It can be easily checked that $\kappa_n(G_2) = 1$ since removal of node 2 partitions the residual graph. Finally, we note that the graph in Figure 2c is 2-connected (removal of nodes 3 and 4 disconnects the residual graph), which is also verified by the fact that $\lambda_2(\mathbf{L}(G_3)) = 2$.

Though not necessary in the context of this paper, it is interesting to note that Kirkland *et al* [16] have recently identified the conditions under which the algebraic connectivity of a graph is exactly equal to its node connectivity. Before stating this result, we need to establish the following notation. For any $N \times N$ matrix $\mathbf{B}$, let the quantity $\mathcal{F}(\mathbf{B})$ be defined as follows:

$$\mathcal{F}(\mathbf{B}) = \frac{1}{2}\left(\max_{1 \le i,j \le N} \sum_{k=1}^{N} \mid \mathbf{B}_{ik} - \mathbf{B}_{jk} \mid\right) \tag{28}$$

The main result of [16] states:

Theorem 5 *Let $G = (\mathcal{N}, \mathcal{E})$ be a non-complete, connected graph on N nodes with $N \ge \kappa_n^2(G)$. Then, $\lambda_2(\mathbf{L}(G)) = \kappa_n(G)$ if and only if:*

$$\kappa_n(G) = \frac{1}{\mathcal{F}\left(\mathbf{L}^{\#}(G)\right)} \tag{29}$$

where $\mathbf{L}^{\#}(G)$ is the group inverse[3] of $\mathbf{L}(G)$. For symmetric matrices, such as a graph laplacian, the group inverse coincides with the more familiar Moore-Penrose inverse or pseudoinverse of a matrix.

We leave it to the reader to verify that the graphs in Figures 2b and 2c do indeed satisfy (29) but that in Figure 2a does not. Another interesting result in [16], which is more of a graph theoretic nature, states:

Theorem 6 *Let $G = (\mathcal{N}, \mathcal{E})$ be a non-complete, connected graph on N nodes. Then, $\lambda_2(\mathbf{L}(G)) = \kappa_n(G)$ if and only if G can be written as $G_1 \vee G_2$, where (a) G_1 is a disconnected graph on $N - \kappa_n(G)$ nodes, and (b) G_2 is a graph on $\kappa_n(G)$ nodes satisfying the spectral condition $\lambda_2(\mathbf{L}(G_2)) \ge 2\kappa_n(G) - N$.*

For all practical purposes, $\kappa_n(G)$ would typically be in the range $1 \le \kappa_n(G) \le 4$. For such a choice of $\kappa_n(G)$, observe that the spectral condition $\lambda_2(\mathbf{L}(G_2)) \ge 2\kappa_n(G) - N$ is trivial and automatically satisfied for all networks with $N \ge 9$, since all eigenvalues of a graph laplacian are known to be non-negative. We now illustrate Theorem 6 with an example.

Consider the graph in Figure 2c whose algebraic connectivity = node connectivity = 2. In Figure 3a and 3b, we show the components G_1 and G_2. Note that:

- G_1 is disconnected and comprises of $N - \kappa_n(G) = 5 - 2 = 3$ nodes, and
- G_2 is a graph on $\kappa_n(G) = 2$ nodes and $\lambda_2(\mathbf{L}(G_2)) = 1 > 2\kappa_n(G) - N = 4 - 5 = -1$.

Since G_1 and G_2 satisfy all the requirements of Theorem 6, we should be able to obtain the original graph in Figure 2c by "joining" G_1 and G_2. Recall from Definition 9 that the join of two graphs G_1 and G_2 is obtained from their union by adding edges from all nodes in G_1 to all nodes in G_2. The union of G_1 and G_2 is shown in Figure 3c and the graph obtained by joining G_1 and G_2 is shown in Figure 3d, which is identical to the graph in Figure 2c. We refer the reader to [16] for details on how to derive the components G_1 and G_2, given a graph G.

While Theorem 6 is essentially an analysis tool which is useful for verifying whether the algebraic connectivity of a graph is equal to its node connectivity, it would be extremely beneficial if its conditions could be used to synthesize graphs exhibiting such property. This is a subject of our future research.

[3] See Chapter 4, Definition 4.12, pp. 118 of [17] for a definition of the group inverse of a matrix.

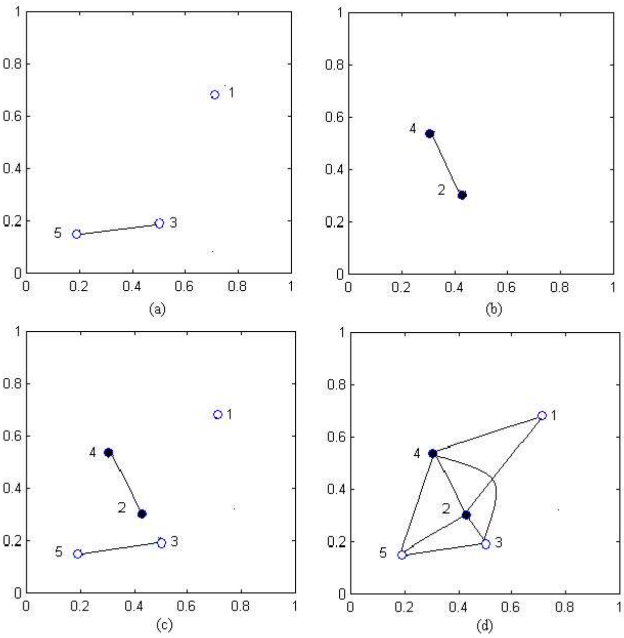

Figure 3: Illustrating how the graph in Figure 2c (whose algebraic connectivity = node connectivity = 2) can be obtained by "joining" two graphs G_1 and G_2. For clarity, we have shown the nodes of G_2 as solid circles. (a) Component G_1. Note that G_1 is disconnected and comprises of $N - \kappa_n(G) = 3$ nodes. (b) Component G_2. As per Theorem 6, this is a graph on $\kappa_n(G) = 2$ nodes and $\lambda_2(\mathbf{L}(G_2)) = 1 > 2\kappa_n(G) - N = -1$. (c) The union of G_1 and G_2, $G_1 \cup G_2$ (d) The join of G_1 and G_2, $G_1 \vee G_2$, obtained by adding the set of edges $\{1 \to 2, 1 \to 4, 3 \to 2, 3 \to 4, 5 \to 2, 5 \to 4\}$ to (c). Note that this graph is identical to Figure 2c.

4.1 Dealing with per-sector maximum power constraint

Minimizing the total transmit power has the effect of limiting the total interference power in the network. Minimizing the maximum transmit power, on the other hand, is especially critical in military applications since it is directly related to the probability of interception/detection. As mentioned in Section 4, the latter criterion can be solved optimally in polynomial time [1]. For $K = 1$, Figure 4 provides an example which illustrates that a minimum power topology (MPT) may not minimize the maximum per-sector transmit power. Consequently, we may want to solve the MPT problem subject to a constraint on the maximum per-sector transmit power.

Let $\hat{Y}$ be the optimal per-sector maximum transmit power obtained after solving the minimax problem. Redefining the set of valid edges as:

$$\mathcal{E} = \{(i \leftrightarrow j) : \ (i,j) \in \mathcal{N}, \ i \neq j, \ \hat{Y} \geq \mathbf{P}_{ij}, \mathbf{P}_{ji}\} \tag{30}$$

in place of (5) and solving the MILP models will yield a constrained minimum power topology such that the per-sector transmit power of all nodes is not greater than $\hat{Y}$.

5 Topology Construction Heuristic

In this section, we describe a heuristic algorithm for constructing a power efficient K-connected topology for $K \geq 2$. The heuristic is similar to Kruskal's algorithm for the Minimum Spanning Tree problem, with three salient differences:

- unlike Kruskal's algorithm which minimizes the sum of *edge weights*, our algorithm attempts to minimize the sum of *node-sector weights*, the weight of node i, sector s, being defined as the maximum weight of all edges incident on that sector (20).
- whereas Kruskal's algorithm chooses the minimum weight edge at every iteration from the same set of edge weights, our algorithm implements an *incremental cost mechanism* to identify the edge to be chosen at any

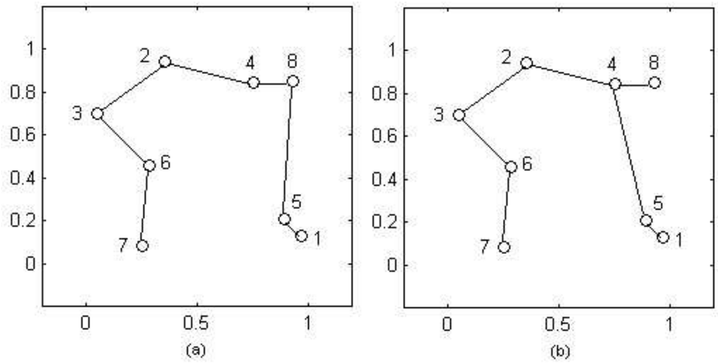

Figure 4: (a) Optimal topology ($K = 1$) minimizing the maximum transmit power. The total transmit power is 1.61 and the maximum power is 0.41, at nodes 5 and 8. (b) Optimal topology ($K = 1$) minimizing the total transmit power. The total transmit power is 1.50 and the maximum power is 0.43, at nodes 4 and 5.

iteration[4]. Edges are added sequentially using an incremental cost mechanism and connectivity of the augmented graph is checked using the λ_2 criterion (22).

In terms of time complexity, the critical component is computation of λ_2. While polynomial time algorithms (see Section 8.3 of [18]) do exist for spectral decomposition of a real, symmetric matrix, it is desirable that such computations be minimized, especially for large N. As shown subsequently, the eigenvalue interlacing properties of Theorem 4 can be invoked to decide whether it is necessary to recompute λ_2 of the laplacian after addition of every single single edge or whether a block of edges can be added before λ_2 is recomputed.

- unlike Kruskal's algorithm, there is no cycle avoidance condition during the edge selection process.

Before describing the algorithm, we establish the following notation:

t = iteration number
$\mathbf{Y}^t$ = $N \times S$ matrix of node-sector powers after iteration t
$\mathbf{\Theta}$ = $N \times N$ sector matrix

The $[i, j]th$ element of $\mathbf{\Theta}$ specifies the sector in which node j is located w.r.t node i.

Next, we illustrate the incremental cost mechanism with an example. In Figure 5, assume that nodes 2, 3, 4 and

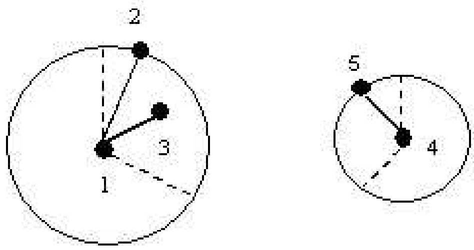

Figure 5: Illustrating the concept of *incremental cost* of choosing an edge. Assume that nodes 2, 3, 4 and 5 are located in sector 1 w.r.t node 1 and nodes 5, 2, 3 and 1 are located in sector 3 w.r.t node 4. Since node 1 is maintaining bidirected edges with 2 and 3, the transmit power required at its sector 1 antenna is given by: $\mathbf{Y}_{1,1} = max(\mathbf{P}_{12}, \mathbf{P}_{13}) = \mathbf{P}_{12}$. Similarly, since node 4 is currently maintaining a bidirected edge with 5, the transmit power level of its sector 3 antenna is given by: $\mathbf{Y}_{4,3} = \mathbf{P}_{45}$. The incremental cost of choosing the edge ($1 \leftrightarrow 4$) is defined as the incremental transmit power support required at node 1's sector 1 antenna + the incremental transmit power support required at node 4's sector 3 antenna = $max(0, \mathbf{P}_{1,4} - \mathbf{Y}_{1,1}) + max(0, \mathbf{P}_{4,1} - \mathbf{Y}_{4,3})$.

[4]The incremental cost criterion was proposed by Wieselthier *et al* [10] in the context of broadcast/multicast routing in wireless networks.

5 are located in sector 1 w.r.t node 1 and nodes 5, 2, 3 and 1 are located in sector 3 w.r.t node 4. Since node 1 is currently maintaining bidirected edges with nodes 2 and 3, the transmit power level of its sector 1 antenna is given by: $\mathbf{Y}_{1,1} = max(\mathbf{P}_{12}, \mathbf{P}_{13}) = \mathbf{P}_{12}$. Similarly, since node 4 is currently maintaining a bidirected edge with 5, the transmit power level of its sector 3 antenna is given by: $\mathbf{Y}_{4,3} = \mathbf{P}_{45}$. The incremental cost of choosing the edge ($1 \leftrightarrow 4$) is defined as the incremental transmit power support required at node 1's sector 1 antenna + the incremental transmit power support required at node 4's sector 3 antenna = $(\mathbf{P}_{1,4} - \mathbf{Y}_{1,1}) + (\mathbf{P}_{4,1} - \mathbf{Y}_{4,3})$.

In general, the incremental cost of choosing an edge $(i \leftrightarrow j)$ at iteration t, $IC(i \leftrightarrow j)$, is defined as the sum of the incremental power supports required at nodes i and j, denoted by $IC(i)$ and $IC(j)$.

$$\begin{aligned} IC(i \leftrightarrow j) &= IC(i) + IC(j) \\ &= max(0, \mathbf{P}_{ij} - \mathbf{Y}^{t-1}_{i,\Theta_{ij}}) + max(0, \mathbf{P}_{ji} - \mathbf{Y}^{t-1}_{j,\Theta_{ji}}) \end{aligned} \tag{31}$$

For any t, the edge which incurs the *minimum incremental cost* is chosen from the set of edges which have not yet been selected. Ties, if any, are broken arbitrarily.

Suppose that the minimum incremental cost edge at any iteration t is ($m \leftrightarrow n$). The node-sector power matrix at the end of iteration t is then updated as follows:

$$\mathbf{Y}^t_{is} := \begin{cases} max(\mathbf{Y}^{t-1}_{is}, \mathbf{P}_{mn}), & \text{if } (i,s) = (m, \Theta_{mn}) \\ max(\mathbf{Y}^{t-1}_{is}, \mathbf{P}_{nm}), & \text{if } (i,s) = (n, \Theta_{nm}) \\ \mathbf{Y}^{t-1}_{is}, & \text{otherwise} \end{cases} \tag{32}$$

We now proceed to describe the algorithm in greater detail. For clarity, we divide the algorithm into two phases, depending on the necessity (or lack thereof) for computing the second eigenvalue of the laplacian.

In Phase 1, edges are added sequentially according to the minimum incremental cost criterion until the minimum node degree is greater than or equal to the desired node connectivity parameter, K. No eigendecompositions of the laplacian matrices are necessary in this phase since the minimum node degree must be at least equal to K for a graph to be K-connected (see Theorem 2, equation (12)). A high level description of Phase 1 of the algorithm is provided in Figure 6.

---

1. Set $t = 0$.
2. Set $\mathbf{Y}^t = 0$.
3. Set $G = \emptyset$.
4. **while** ($\min_i(deg_i(G)) < K$)
   - Set $t = t + 1$;
   - Compute the set of $candidate\_edges = \mathcal{E} \setminus G$
   - Select the edge, $i \leftrightarrow j$, from $candidate\_edges$ which incurs the minimum incremental cost (31)
   - $G = \{G \cup (i \leftrightarrow j)\}$ /* Append $(i \leftrightarrow j)$ to G */
   - Update $\mathbf{Y}^t$ as in (32).
   - Set $t = t + 1$;

   end while

Figure 6: Phase 1 of the topology construction heuristic. Edges are repeatedly chosen according to the minimum incremental cost criterion until the minimum node degree is greater than or equal to the desired node connectivity parameter, K. No eigendecomposition of the laplacian matrix is necessary in this phase.

Phase 2 of the algorithm is similar to Phase 1 except that an eigendecomposition of the laplacian matrix is necessary at the end of each iteration to check if the graph connectivity condition (22) is satisfied. Also, unlike Phase 1 where only one edge is chosen at each iteration, it may be possible to add multiple edges at each Phase 2 iteration. The conditions under which this is admissible are discussed below.

Let $\{\lambda_1(\mathbf{L}(G)) \leq \lambda_2(\mathbf{L}(G)) \leq \cdots \leq \lambda_N(\mathbf{L}(G))\}$ be the ordered eigenspectrum of the laplacian of the graph G. If we add one edge to G, thereby obtaining the graph G', Theorem 4 tells us that:

$$\lambda_2(\mathbf{L}(G')) \leq \lambda_3(\mathbf{L}(G)) \text{ and } \lambda_3(\mathbf{L}(G')) \leq \lambda_4(\mathbf{L}(G)) \tag{33}$$

If $\lambda_3(\mathbf{L}(G)) \leq K-1$, we do not need to compute $\lambda_2(\mathbf{L}(G'))$ since

$$\lambda_2(\mathbf{L}(G')) \leq \lambda_3(\mathbf{L}(G)) \leq K-1$$

and therefore the graph G' cannot be K-connected.

Next, let us consider the case when an edge is added to G', resulting in the graph G''. Using Theorem 4, we have:

$$\lambda_2(\mathbf{L}(G'')) \leq \lambda_3(\mathbf{L}(G')) \tag{34}$$

If we know that $\lambda_4(\mathbf{L}(G)) \leq K-1$, we do not need to compute $\lambda_2(\mathbf{L}(G''))$ since

$$\lambda_2(\mathbf{L}(G'')) \leq \lambda_3(\mathbf{L}(G')) \leq \lambda_4(\mathbf{L}(G)) \leq K-1$$

where the second inequality follows from (33). Consequently, the graph G'' cannot be K-connected.

Following the above reasoning, it is easy to see that if m ($3 \leq m \leq N$) is the index such that

$$\lambda_1(\mathbf{L}(G)) \leq \cdots \leq \lambda_m(\mathbf{L}(G)) \leq K-1 \tag{35}$$

and

$$K-1 < \lambda_{m+1}(\mathbf{L}(G)) \leq \cdots \leq \lambda_N(\mathbf{L}(G)) \tag{36}$$

are satisfied, then the maximum number of edges that can be added to G such that the second eigenvalue of the laplacian of the resulting graph is less than K is equal to $m-2$.

Note that, if $m=2$, $i.e.$, if $\lambda_2(\mathbf{L}(G)) \leq K-1$ and $\lambda_3(\mathbf{L}(G)) > K-1$, we cannot say with certainty whether addition of an edge will achieve the desired connectivity or not. Consequently, an eigendecomposition is necessary with the addition of every single edge to G.

We therefore have the following lemma:

Lemma 1 *Let* $\{0 = \lambda_1(\mathbf{L}(G)) \leq \lambda_2(\mathbf{L}(G)) \leq \cdots \leq \lambda_N(\mathbf{L}(G))\}$ *be the ordered eigenspectrum of the laplacian of the graph* G*. Let* m *(*$2 \leq m \leq N$*) be the index such that (35) and (36) are satisfied. Then, the maximum number of edges that can be added to* G*, such that the second eigenvalue of the laplacian of the resulting graph is less than or equal to* $K-1$*, is equal to* $\max(1, m-2)$.

With the above lemma in place, we are now in a position to complete our discussion of Phase-2 of the algorithm. Let G_t be the graph obtained after iteration t. Also, let $\{\lambda_1(\mathbf{L}(G_t)) \leq \lambda_2(\mathbf{L}(G_t)) \leq \cdots \leq \lambda_N(\mathbf{L}(G_t))\}$ be the ordered eigenspectrum of the laplacian of G_t and m the index which satisfies (35) and (36). If $\lambda_2(\mathbf{L}(G_t)) > K-1$, we have a K-connected topology and the algorithm terminates. If not, we first define a set of candidate edges, $candidate\_edges = \mathcal{E} \setminus G_t$, where $\mathcal{E}$ is as defined in (5) or (30). Next,

- if $m=2$, the minimum incremental cost edge (31), say $i \leftrightarrow j$, is chosen from the set $candidate\_edges$ and appended to G_t.
- if $3 \leq m \leq N$, $(m-2)$ minimum incremental cost edges are chosen sequentially and appended to G_t.

Define G_{t+1} to be the graph obtained after the edge(s) have been appended to G_t. If $\lambda_2(\mathbf{L}(G_{t+1})) > K-1$, the algorithm terminates. Otherwise, the procedure is repeated on G_{t+1}. Figure 7 summarizes Phase 2 of the topology construction algorithm.

From a computational aspect, it should be noted that, in general, deciding whether multiple edges can be added at iteration t requires a complete eigendecomposition of the laplacian of the graph obtained at iteration $t-1$. However, from the eigenvalue interlacing theorem (15), it follows that if multiple edges have been added till iteration $t'-1$ and one edge is added at iteration t', all subsequent iterations will also add only one edge. Consequently, a full eigendecomposition is not required for all iterations $t \geq t'$. It may be computationally cheaper to determine only the second eigenvalue of the laplacian in terms of the maximum eigenvalue of the laplacian of the complement graph (14), which can be computed using the power method [18].

1. Set $t = 0$;
2. Set G = 1-connected topology obtained after Phase-1;
3. Compute the eigenvalues of $\mathbf{L}(G)$.
4. Set $\mathbf{Y}^0$ = node-sector power matrix corresponding to G;
5. **while** $(\lambda_2(\mathbf{L}(G)) \leq K - 1)$
   Increment $t = t + 1$;
   Let m be the index satisfying (35) and (36).
   if $(m == 2)$
   • Compute the set of $candidate\_edges = \mathcal{E} \setminus G$.
   • Select the edge, $i \leftrightarrow j$, from $candidate\_edges$ which incurs the minimum incremental cost (31).
   • $G = \{G \cup (i \leftrightarrow j)\}$; /* Append $(i \leftrightarrow j)$ to G */
   • Update $\mathbf{Y}^t$ as in (32).
   elseif $(3 \leq m \leq N)$
   Initialize $Y^0_{temp} = \mathbf{Y}^{t-1}$;
   for $(s = 1 : m - 2)$
   • Compute the set of $candidate\_edges = \mathcal{E} \setminus G$;
   • Select the edge, $i \leftrightarrow j$, from $candidate\_edges$ which incurs the minimum incremental cost (31).
   • $G = \{G \cup (i \leftrightarrow j)\}$;
   • Update $\mathbf{Y}^s_{temp}$ as in (32).
   end for
   Assign $\mathbf{Y}^t = \mathbf{Y}^s_{temp}$;
   end if
   Compute the eigenvalues of $\mathbf{L}(G)$.
   end while
6. Cost of the final K-connected topology, G, is equal to $\sum_{i,s} \mathbf{Y}^t_{is}$.

Figure 7: Phase 2 of the topology construction heuristic. An eigendecomposition of the laplacian matrix is necessary at the end of each iteration to check if the graph connectivity condition (22) is satisfied. Also, unlike Phase 1 where only one edge is chosen at each iteration, it may be possible to add multiple edges at each Phase 2 iteration.

6 Topology Improvement Heuristic

As mentioned previously, the K-connected topology obtained after Phase-2 can include several non-essential edges, deletion of which *may* result in a reduction of the total power cost, without affecting the algebraic connectivity. Unfortunately, identifying such edges is not straightforward and the eigenvalue interlacing theorem, which we were able to use in Phase-2 of the topology construction algorithm to add a block of edges at each iteration and thereby reduce the number of times an eigendecomposition needs to be performed, does not help. To see why, let G be a K-connected graph ($\Rightarrow \lambda_2(\mathbf{L}(G)) > K - 1$) and G' the graph obtained by deleting any edge e from G. Using (15), we can write:

$$0 = \lambda_1(\mathbf{L}(G')) = \lambda_1(\mathbf{L}(G)) \leq \lambda_2(\mathbf{L}(G')) \leq \lambda_2(\mathbf{L}(G)), \quad \lambda_2(\mathbf{L}(G)) > K - 1 \tag{37}$$

It is clear from (37) that the only inference we can make on the second eigenvalue of $\mathbf{L}(G')$, without actually computing it, is that $\lambda_2(\mathbf{L}(G')) \geq 0$, which is trivial. The upshot of this is that, connectivity needs to be checked every time an edge is deleted. Computationally, therefore, the topology improvement phase can be significantly more expensive than the topology construction phase discussed in Section 5.

Given the topology from Section 5, we first order the edges based on their weights *relative to the topology*, as discussed subsequently. Next, we scan them in descending order (starting with the edge that has the highest finite relative weight and finishing with the edge that has the smallest relative weight) to check whether K-connectivity is maintained if a certain edge is deleted. If so, that edge is deleted and the scan is repeated on the new graph, with a new set of relative edge weights. The algorithm terminates when no edge can be deleted without adversely affecting the connectivity.

For any sectored antenna network, the weight of the edge, $i \leftrightarrow j$, relative to a topology G, is denoted by $\mathbf{W}_{ij}(G)$

and defined as follows:

$$\mathbf{W}_{ij}(G) = \mathbf{P}_{ij} \cdot \delta(\mathbf{P}_{ij}, \mathbf{Y}_{i,\theta_{ij}}) + \mathbf{P}_{ji} \cdot \delta(\mathbf{P}_{ji}, \mathbf{Y}_{j,\theta_{ji}}) \quad (38)$$
$$= \mathbf{P}_{ij} \cdot [\delta(\mathbf{P}_{ij}, \mathbf{Y}_{i,\theta_{ij}}) + \delta(\mathbf{P}_{ij}, \mathbf{Y}_{j,\theta_{ji}})] \quad (39)$$

where

- θ_{ij} is the sector in which node j is located w.r.t node i,
- $\mathbf{Y}_{i,\theta_{ij}}$ is the transmitter power level corresponding to sector θ_{ij} of node i in topology G, and
- $\delta(\mathbf{P}_{ij}, \mathbf{Y}_{i,\theta_{ij}})$ is the Kronecker delta function:

$$\delta(\mathbf{P}_{ij}, \mathbf{Y}_{i,\theta_{ij}}) = \begin{cases} 1; & \text{if } \mathbf{P}_{ij} = \mathbf{Y}_{i,\theta_{ij}} \\ 0; & \text{otherwise} \end{cases} \quad (40)$$

Note that we have used the symmetric property of the matrix $\mathbf{P}$ in obtaining (39) from (38). For omnidirectional antennas, all θ_{ij}'s are equal and therefore redundant. In other words, $\mathbf{Y}_{i,\theta_{ij}} = \mathbf{Y}_i$.

We now illustrate with an example the concept of relative weights of edges. Consider the omnidirectional topology in Figure 8. The transmit power costs of the edges are shown in the top left hand corner of the figure and the node costs corresponding to the topology are shown in the bottom right hand corner. For example, $Y_2 = \max(\mathbf{P}_{21}, \mathbf{P}_{23}, \mathbf{P}_{24}, \mathbf{P}_{25}) = \mathbf{P}_{21} = 5$. The total transmit power cost is 18 units.

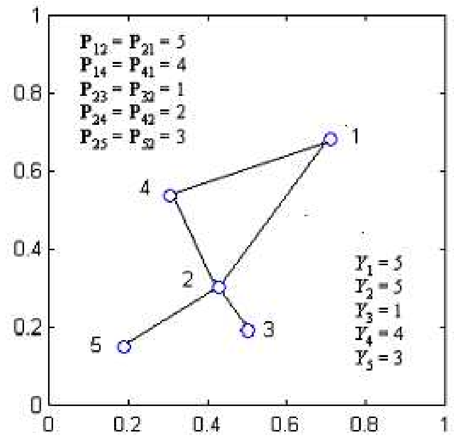

Figure 8: Example topology for illustrating the concept of relative weights of edges. Assume that all nodes have omnidirectional antennas. The transmit power costs of the edges are shown in the top left hand corner of the figure and the node costs corresponding to the topology are shown in the bottom right hand corner.

For the topology in Figure 8, the relative weight of $1 \leftrightarrow 4$ is:

$$\mathbf{W}_{14}(G) = \mathbf{P}_{14} \cdot [\delta(\mathbf{P}_{14}, \mathbf{Y}_1) + \delta(\mathbf{P}_{14}, \mathbf{Y}_4)]$$
$$= 4 \cdot [0 + 1] = 4$$

Following a similar procedure, the relative weights of the other edges in Figure 8 are found to be:

$$\mathbf{W}_{12}(G) = 10,\ \mathbf{W}_{24}(G) = 0,\ \mathbf{W}_{25}(G) = 3,\ \mathbf{W}_{23}(G) = 1$$

Observe that the sum of the relative edge weights is equal to the sum of the node transmit power levels. Interestingly, even though $\mathbf{P}_{24} = 2$, the relative weight of the edge $2 \leftrightarrow 4$ is zero since the transmit power levels of both its end nodes are greater than 2 and therefore it does not contribute to the cost at either node. Deleting this edge, therefore, would not reduce the total power cost. For our example, the ordered edge list based on relative weights is:

$$(1 \leftrightarrow 2), (1 \leftrightarrow 4), (2 \leftrightarrow 5), (2 \leftrightarrow 3), (2 \leftrightarrow 4)$$

For any topology, let EP be the number of edges with strictly positive relative weights. Since reduction in topology cost would occur only if one of these edges can be deleted without affecting the connectivity, each scan through the ordered edge list can be limited to the first EP edges.

A further refinement can be made by making use of the property that the node connectivity of a graph is upper bounded by the minimum node degree (12). Clearly, an edge $i \leftrightarrow j$ in a graph G can be a candidate for deletion if:

$$deg_i(G) \geq K + 1 \text{ and } deg_j(G) \geq K + 1 \tag{41}$$

If either of these two conditions is not satisfied, deleting the edge would definitely result in a drop of the graph connectivity below K.

Figure 9 provides a high level description of the topology improvement heuristic.

---

1. Set G = K-connected topology obtained from Section 5;
2. Set $G_{new} = \emptyset$;
3. Set $flag = 1$;
4. **while** (flag)
   - Set $edge\_list$ = edge list of G arranged in descending order of relative weights;
   - Let EP be the number of edges with strictly positive relative weights.
   - Set $G_{new} = G$;

   for ($m = 1 : EP$)
     if (the end nodes of $edge\_list(m)$ satisfy eqn. (41))
       /* Delete $edge\_list(m)$ from G and assign it to G' */
       Set $G' = G \setminus edge\_list(m)$;
       if $(\lambda_2(\mathbf{L}\,(G')) > K - 1)$
         /* Delete the m-th edge in $edge\_list$ from G_{new} */
         • $G_{new} = \{G_{new} \setminus edge\_list(m)\}$;
         • **break** /* Break out of the **for** loop */
       end if
     end if
   end for
   if ($G_{new} == G$)
     /* All edges with positive relative weight scanned; no cost improvement. Terminate improvement phase. */
     $flag = 0$;
   else
     • Set $G = G_{new}$;
     • Set $G_{new} = \emptyset$;
   end if
   end while
5. Compute the cost of the improved K-connected topology, G, using equation (24).

Figure 9: High level description of a topology improvement heuristic. It is used to identify and delete non-essential edges from the graph obtained after the topology construction phase. Deletion of these edges may result in a reduction of the total power cost, without affecting the algebraic connectivity.

7 An example

We now provide an example to illustrate the topology construction and improvement heuristics. Consider the 7-node network in Figure 10 and suppose that we seek to construct a power efficient 2-connected topology ($K = 2$). Assume that all nodes are equipped with 3-sector antennas. The link cost matrix (7) for $\alpha = 2$ and the sector matrix of the network are as follows:

$$\mathbf{C} = \begin{bmatrix} \infty & 0.9523 & 1.3698 & 1.0026 & 0.3997 & 0.9253 & 0.1354 \\ 0.9523 & \infty & 0.0885 & 0.5521 & 0.6987 & 0.1028 & 1.5038 \\ 1.3698 & 0.0885 & \infty & 1.0691 & 1.2664 & 0.0600 & 2.1403 \\ 1.0026 & 0.5521 & 1.0691 & \infty & 0.1733 & 1.0580 & 1.0263 \\ 0.3997 & 0.6987 & 1.2664 & 0.1733 & \infty & 1.0527 & 0.3561 \\ 0.9253 & 0.1028 & 0.0600 & 1.0580 & 1.0527 & \infty & 1.6366 \\ 0.1354 & 1.5038 & 2.1403 & 1.0263 & 0.3561 & 1.6366 & \infty \end{bmatrix} \tag{42}$$

$$\mathbf{\Theta} = \begin{bmatrix} - & 1 & 1 & 1 & 2 & 1 & 2 \\ 3 & - & 1 & 2 & 2 & 3 & 2 \\ 2 & 2 & - & 2 & 2 & 3 & 2 \\ 3 & 3 & 3 & - & 3 & 3 & 3 \\ 3 & 1 & 1 & 1 & - & 1 & 3 \\ 2 & 2 & 1 & 2 & 2 & - & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 & - \end{bmatrix} \tag{43}$$

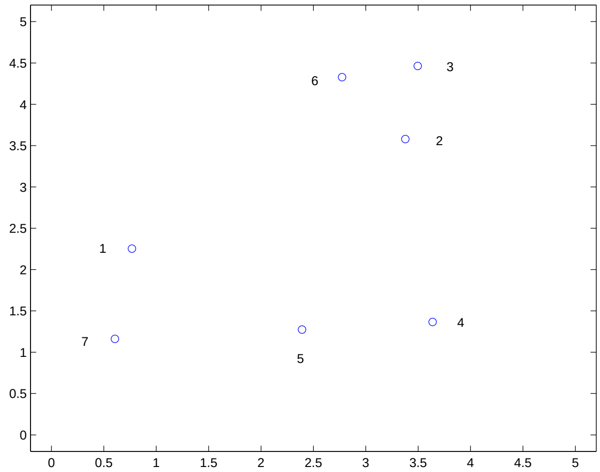

Figure 10: Example 7-node topology.

7.1 Topology construction phase

For Phase 1 of the construction phase, we initialize the topology and the node-sector power matrix as follows:

$$G = \emptyset \text{ and } \mathbf{Y}^0 = \mathbf{0}$$

Iteration 1: It is obvious that the first edge selected is $(3 \leftrightarrow 6)$ which has the minimum cost in (42). We now update $\mathbf{Y}$. Since node 6 is located in sector 3 w.r.t node 3 ($\theta_{36} = 3$ in (43)), we update the $[3, 3]th$ element of $\mathbf{Y}^1$ as follows:

$$\mathbf{Y}^1_{33} = max(\mathbf{Y}^0_{33}, \mathbf{C}_{36}) = 0.0600$$

Next, we update the $[6, 1]th$ element of $\mathbf{Y}^1$ since node 3 is located in sector 1 w.r.t node 6:

$$\mathbf{Y}^1_{61} = max(\mathbf{Y}^0_{61}, \mathbf{C}_{63}) = 0.0600$$

Updating both these elements ensures that the transmit powers of nodes 3 and 6 are adequate to support the bidirectional link ($3 \leftrightarrow 6$).The modified node-sector power matrix is therefore:

$$\mathbf{Y}^1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.0600 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.0600 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (44)$$

The partial topology at the end of the first iteration is:

$$G = \{(3 \leftrightarrow 6)\} \quad (45)$$

Iteration 2: The candidate edge list for iteration 2 is:

$$candidate\_edges = \begin{bmatrix} - & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & - & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & - & 1 & 1 & - & 1 \\ 1 & 1 & 1 & - & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & - & 1 & 1 \\ 1 & 1 & - & 1 & 1 & - & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & - \end{bmatrix} \quad (46)$$

A '1' in the $[i, j]^{th}$ element (and $[j, i]^{th}$ since we are dealing with bidirected edges) of (46) implies that the edge $i \leftrightarrow j$ is a candidate edge. We now compute the incremental costs of choosing the edges in (46). By the sectored wireless advantage property (see Figure 1), only those nodes which are located in the same sector as node 6 (node 3) w.r.t node 3 (node 6) will require reduced incremental power support. A quick check of (43) reveals that there is no other node co-located in the same sector as node 6 (node 3) w.r.t node 3 (node 6). The matrix of incremental costs is therefore:

$$incremental\_costs = \begin{bmatrix} \infty & 0.9523 & 1.3698 & 1.0026 & 0.3997 & 0.9253 & 0.1354 \\ 0.9523 & \infty & 0.0885 & 0.5521 & 0.6987 & 0.1028 & 1.5038 \\ 1.3698 & 0.0885 & \infty & 1.0691 & 1.2664 & - & 2.1403 \\ 1.0026 & 0.5521 & 1.0691 & \infty & 0.1733 & 1.0580 & 1.0263 \\ 0.3997 & 0.6987 & 1.2664 & 0.1733 & \infty & 1.0527 & 0.3561 \\ 0.9253 & 0.1028 & - & 1.0580 & 1.0527 & \infty & 1.6366 \\ 0.1354 & 1.5038 & 2.1403 & 1.0263 & 0.3561 & 1.6366 & \infty \end{bmatrix} \quad (47)$$

It is clear from (47) that the edge which incurs minimum incremental cost is $2 \leftrightarrow 3$. The updates in this iteration are therefore:

$$G = \{(3 \leftrightarrow 6), (2 \leftrightarrow 3)\} \quad (48)$$

$$\mathbf{Y}^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0.0885 & 0 & 0 \\ 0 & 0.0885 & 0.0600 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.0600 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (49)$$

Note that the $[2, 1]^{th}$ and $[3, 2]^{th}$ elements of $\mathbf{Y}$ are updated since node 3 (node 2) is located in sector 1 (sector 2) w.r.t node 2 (node 3).

Following a similar procedure, it can be verified that the topology obtained after Phase 1 of the construction phase (which terminates when the minimum node degree is equal to 2) is as shown in Figure 11.

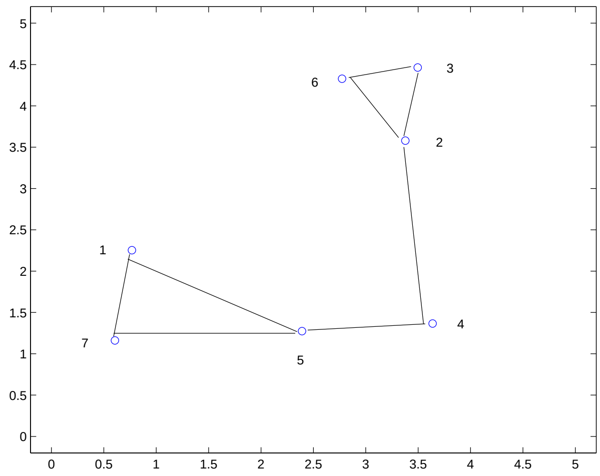

Figure 11: Topology obtained after Phase 1 of the construction heuristic.

The corresponding node-sector power matrix is:

$$\mathbf{Y} = \begin{bmatrix} 0 & 0.3997 & 0 \\ 0.0885 & 0.5521 & 0.1028 \\ 0 & 0.0885 & 0.0600 \\ 0 & 0 & 0.5521 \\ 0.1733 & 0 & 0.3997 \\ 0.0600 & 0.1028 & 0 \\ 0.3561 & 0 & 0 \end{bmatrix} \tag{50}$$

We now proceed to Phase 2 of the construction heuristic with an eigendecomposition of the laplacian matrix corresponding to the graph in Figure 11, which is:

$$[0,\ 0.2679,\ 1.5858,\ 3.0000,\ 3.0000,\ 3.7321,\ 4.4142]$$

Since $\lambda_2 = 0.2679 \leq K - 1 = 1$ and $\lambda_3 = 1.5858 > K - 1 = 1$, the index m in Lemma 1 is equal to 2 $\Rightarrow$ the number of edges which can be added at the first iteration (and all subsequent ones) of Phase 2 is 1. The edges which are selected in order are $2 \leftrightarrow 5$, $4 \leftrightarrow 7$, $4 \leftrightarrow 6$ and $5 \leftrightarrow 6$, leading to the topology shown in Figure 12. The ordered eigenspectrum of the laplacian corresponding to the topology in Figure 12 is:

$$[0,\ 1.1442,\ 2.5858,\ 3.6784,\ 5.0000,\ 5.4142,\ 6.1774]$$

Since $\lambda_2 = 1.1442 > K - 1 = 1$, this topology is 2-connected. The corresponding node-sector power matrix is

$$\mathbf{Y} = \begin{bmatrix} 0 & 0.3997 & 0 \\ 0.0885 & 0.6987 & 0.1028 \\ 0 & 0.0885 & 0.0600 \\ 0 & 0 & 1.0580 \\ 1.0527 & 0 & 0.3997 \\ 0.0600 & 1.0580 & 0 \\ 1.0263 & 0 & 0 \end{bmatrix} \tag{51}$$

and the topology cost is $\sum_{i,s} \mathbf{Y}_{is} = 6.0928$ units.

7.2 Topology improvement phase

Using (39), the link cost matrix (42) and the node-sector power matrix (52), the sorted edge list corresponding to the topology in Figure 12, based on strictly positive relative weights, is:

$$4 \leftrightarrow 6,\ 5 \leftrightarrow 6,\ 4 \leftrightarrow 7,\ 1 \leftrightarrow 5,\ 2 \leftrightarrow 5,\ 2 \leftrightarrow 3,\ 3 \leftrightarrow 6,\ 2 \leftrightarrow 6$$

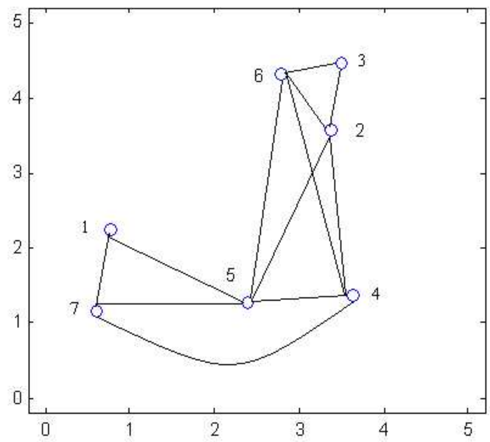

Figure 12: Topology obtained after Phase 2 of the construction heuristic. The second laplacian eigenvalue is 1.1442, which is greater than $K - 1 = 1$. Consequently, the topology is 2-connected. The cost of the topology is 6.0928 units.

The first edge which is considered for deletion is therefore $4 \leftrightarrow 6$. It can be verified that the second eigenvalue of the laplacian with the edge $4 \leftrightarrow 6$ deleted is 1.0148, greater than $K - 1 = 1$. Consequently, it can be removed from the topology in Figure 12.

It can also be verified that there is no other edge with positive relative weight which can be deleted without adversely affecting the connectivity. The final topology is therefore as shown in Figure 13. The corresponding node-sector power matrix is:

$$\mathbf{Y} = \begin{bmatrix} 0 & 0.3997 & 0 \\ 0.0885 & 0.6987 & 0.1028 \\ 0 & 0.0885 & 0.0600 \\ 0 & 0 & 1.0263 \\ 1.0527 & 0 & 0.3997 \\ 0.0600 & 1.0527 & 0 \\ 1.0263 & 0 & 0 \end{bmatrix} \quad (52)$$

and the topology cost is $\sum_{i,s} \mathbf{Y}_{is} = 6.0559$, 0.0369 units less than the cost of the topology obtained after the construction phase.

8 Simulation Results

We have conducted a performance study of our heuristic method on 100 randomly generated networks of sizes ranging from 20 to 100 nodes randomly distributed in a 5×5 grid. Each node is assumed to have a 3-sector antenna, $i.e.$, $S = 3$ in (3). The exponent 'α' in (3) was chosen to be 2 and the parameter P^{max} was set experimentally so that the average algebraic connectivities of the reachability graphs (see Definition 10, Section 2) are between 5.5 and 6.5. The choices of P^{max} for different network sizes and the resulting average number of edges in the reachability graphs and average algebraic connectivities are given in Table 1. Finally, the normalized residual battery capacity of all nodes was chosen to be equal to 1 for all simulations, $i.e.$, $C_i(t) = 1, \forall i$.

8.1 Performance for $K = 1$

The algorithm described in this paper is applicable for any desired K. However, for the special case of $K = 1$ (simple connectivity), an alternate algorithm was proposed by one of the authors in [9]. We first compare the performance of the algorithm described here w.r.t the algorithm in [9]. Table 2 provides a comparison of the topology costs (averaged over 100 trials) obtained using the two algorithms. It can be seen from the table that both the algorithms provide

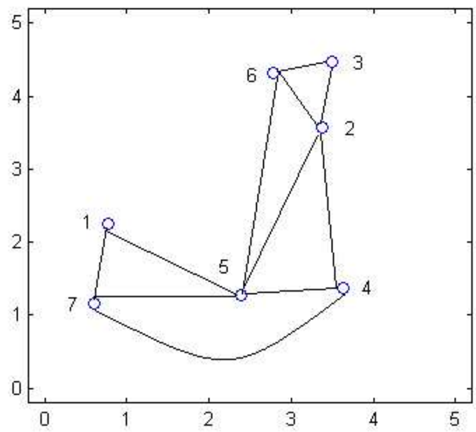

Figure 13: Topology obtained after the improvement phase. Note that the edge $4 \leftrightarrow 6$ has been deleted. The second laplacian eigenvalue is 1.0148, which is greater than $K - 1 = 1$. Consequently, the topology is still 2-connected. Cost of the topology is 6.0559, 0.0369 units less than the topology in Figure 12.

Table 1: Choices of P^{max} and corresponding number of edges and algebraic connectivities of the reachability graphs, averaged over 100 trials.

N	P^{max}	Avg. E	Avg. λ_2 of reachability graph
20	1.20	135	6.04
40	0.80	435	6.43
60	0.60	779	5.95
80	0.50	1193	5.68
100	0.45	1708	6.07

comparable solutions, as far as the objective function ($i.e.$, $\sum_{i,s} Y_{i,s}$) is concerned. However, the algorithm in [9] is considerably faster than the one proposed in this paper.

Table 2: Comparison of average topology cost using the algorithm described in this paper (second column) and the algorithm described in [9] (third column) for $K = 1$.

N	Avg. topology cost	Avg. topology cost (algorithm in [9])
20	3.10294	3.03164
40	3.26772	3.16378
60	3.20364	3.13420
80	3.08260	3.02084
100	3.02861	2.96715

8.2 Performance for $K \geq 2$

We now present simulation results for $K = 2$, 3 and 4. Table 3a provides a comparison of topology costs, $\sum_{i,s} Y_{i,s}$, after the construction phase ($i.e.$, prior to improvement), averaged over 100 trials. The numbers in boldface represent the ratio of the pre-improvement costs to the post-improvement costs listed in Table 3b. It is clear from this table that:

- for any K, the ratios of pre-improvement to post-improvement costs generally decrease as N increases, and,
- for any N, the ratios of pre-improvement to post-improvement costs decrease as K increases.

Table 3a: Comparison of topology costs, $\sum_{i,s} Y_{i,s}$, after the construction phase ($i.e.$, prior to improvement), averaged over 100 trials. The numbers in $\mathbf{boldface}$ represent the ratio of the pre-improvement costs to the post-improvement costs listed in Table 3b below. For example, for $K = 2$ and $N = 20$, the ratio of the pre-improvement cost to the post-improvement cost is equal to $22.85033/13.73591 = 1.66$.

N	$K = 2$	$K = 3$	$K = 4$
20	22.85033 ($\mathbf{1.66}$)	26.78824 ($\mathbf{1.32}$)	30.21573 ($\mathbf{1.21}$)
40	43.75261 ($\mathbf{1.82}$)	49.35087 ($\mathbf{1.39}$)	53.33181 ($\mathbf{1.25}$)
60	56.90239 ($\mathbf{1.61}$)	61.01041 ($\mathbf{1.27}$)	65.60175 ($\mathbf{1.14}$)
80	66.92112 ($\mathbf{1.48}$)	72.55647 ($\mathbf{1.25}$)	77.96310 ($\mathbf{1.16}$)
100	67.04072 ($\mathbf{1.26}$)	77.99872 ($\mathbf{1.10}$)	83.34039 ($\mathbf{1.06}$)

Table 3b: Comparison of topology costs, $\sum_{i,s} Y_{i,s}$, after the improvement phase, averaged over 100 trials. The numbers in $\mathbf{boldface}$ in the second column represent the ratio of the costs for $K = 2$ to $K = 1$. Similarly, the boldfaced numbers in the third and fourth columns represent the ratio of the costs for $K = 3$ to $K = 1$ and $K = 4$ to $K = 1$. The costs for $K = 1$ appear in Table 2 above. For example, the ratio of the costs for $K = 2$ to $K = 1$, $N = 20$, is equal to $13.73591/3.10294 = 4.43$.

N	$K = 2$	$K = 3$	$K = 4$
20	13.73591 ($\mathbf{4.43}$)	20.32844 ($\mathbf{6.55}$)	24.99365 ($\mathbf{8.05}$)
40	24.02290 ($\mathbf{7.35}$)	35.51164 ($\mathbf{10.87}$)	42.75331 ($\mathbf{13.08}$)
60	35.36220 ($\mathbf{11.04}$)	48.17661 ($\mathbf{15.04}$)	57.41498 ($\mathbf{17.92}$)
80	45.16701 ($\mathbf{14.65}$)	57.85460 ($\mathbf{18.77}$)	67.15750 ($\mathbf{21.79}$)
100	53.23967 ($\mathbf{17.58}$)	70.80666 ($\mathbf{23.38}$)	78.89431 ($\mathbf{26.05}$)

In fact, it can be seen that for $N = 100$ and $K = 4$, the pre-improvement cost is only 1.06 times the post-improvement cost. Consequently, for high enough N and K, the improvement phase of the algorithm described in this paper can be dispensed with altogether, without signifi cantly jeopardizing the solution quality. Doing so would also lead to a signifi cant reduction in computation time.

Table 3b lists the topology costs, $\sum_{i,s} Y_{i,s}$, after the improvement phase, averaged over 100 trials. The numbers in boldface in the second column represent the ratio of the costs for $K = 2$ to $K = 1$. The costs for $K = 1$ appear in Table 2. Similarly, the boldfaced numbers in the third and fourth columns represent the costs for $K = 3$ and $K = 4$, normalized by the cost for $K = 1$. While it is not surprising that the costs increase with K for any N, a closer examination of the normalized cost fi gures reveals that the rate of increase exhibits a diminishing trend as K increases. For example, for $N = 60$, augmenting a simple connected topology to a biconnected topology entails an approximately 11-fold increase in cost. However, augmenting it to 3 or 4-connected topologies entails "only" 15 or 18-fold increases in cost. In highly hostile environments, therefore, it may be possible to obtain higher degrees of fault-tolerance at a relatively small increase in topological cost.

Next, we compare the maximum node-sector transmit powers, $i.e.$, $\max_{i,s} Y_{i,s}$, of the optimized topologies (optimized for $\sum_{i,s} Y_{i,s}$), before and after the improvement phase. As mentioned previously, this criterion is specially important in military applications since it directly affects the probability of interception/detection. Table 4a provides a comparison of the maximum node-sector transmit powers after the construction phase. The numbers in boldface represent the ratios of pre-improvement maximum node-sector transmit powers to post-improvement maximum node-sector transmit powers listed in Table 4b. Similar to our observations w.r.t Table 3a, it can be seen that:

- for any K, the ratios of pre-improvement to post-improvement maximum node-sector transmit powers generally decrease as N increases, and,
- for any N, the ratios of pre-improvement to post-improvement maximum node-sector transmit powers decrease as K increases.

Observe that, for $N = 100$ and $K = 4$, the pre-improvement maximum node-sector transmit power is only 1.05 times the post-improvement maximum node-sector transmit power.

Table 4a: Comparison of maximum node-sector transmit power, $\max_{i,s} Y_{i,s}$, in the topologies after the construction phase ($i.e.$, prior to improvement), averaged over 100 trials. The numbers in $\mathbf{boldface}$ represent the ratio of pre-improvement maximum node-sector transmit power to post-improvement maximum node-sector transmit power listed in Table 4b below. For example, for $K = 2$ and $N = 20$, the ratio of pre-improvement maximum node-sector transmit power to post-improvement maximum node-sector transmit power is equal to $0.98815/0.62411 = 1.58$.

N	$K = 2$	$K = 3$	$K = 4$
20	0.98815 (**1.58**)	1.10381 (**1.30**)	1.15732 (**1.16**)
40	0.77188 (**1.75**)	0.79400 (**1.32**)	0.79661 (**1.18**)
60	0.59413 (**1.61**)	0.59772 (**1.26**)	0.59840 (**1.14**)
80	0.49911 (**1.39**)	0.49922 (**1.22**)	0.49935 (**1.15**)
100	0.44662 (**1.35**)	0.44951 (**1.11**)	0.44955 (**1.05**)

Table 4b: Comparison of maximum node-sector transmit power, $\max_{i,s} Y_{i,s}$, in the topologies after the improvement phase, averaged over 100 trials. The numbers in $\mathbf{boldface}$ in the third column represent the ratio of the maximum node-sector transmit powers for $K = 2$ to $K = 1$. Similarly, the boldfaced numbers in the fourth and fifth columns represent the ratio of the maximum node-sector transmit powers for $K = 3$ to $K = 1$ and $K = 4$ to $K = 1$. For example, the ratio of the maximum node-sector transmit powers for $K = 2$ to $K = 1$, $N = 20$, is equal to $0.62411/0.28293 = 2.21$.

N	$K = 1$	$K = 2$	$K = 3$	$K = 4$
20	0.28293	0.62411 (**2.21**)	0.84808 (**3.00**)	0.99478 (**3.52**)
40	0.21320	0.44033 (**2.07**)	0.60360 (**2.83**)	0.67264 (**3.16**)
60	0.13908	0.36814 (**2.65**)	0.47403 (**3.41**)	0.52498 (**3.77**)
80	0.11203	0.35794 (**3.20**)	0.40761 (**3.64**)	0.43519 (**3.88**)
100	0.07076	0.32994 (**4.66**)	0.40630 (**5.74**)	0.42652 (**6.03**)

Table 4b lists the maximum node-sector transmit powers after the improvement phase. The numbers in boldface in the third column represent the ratio of the maximum node-sector transmit powers for $K = 2$ to $K = 1$. Similarly, the boldfaced numbers in the fourth and fifth columns represent the maximum node-sector transmit powers for $K = 3$ and $K = 4$, normalized by the maximum node-sector transmit power for $K = 1$. As with Table 3b, the most important observation from this table is that, for any N, the rate at which the maximum node-sector power for $K \geq 2$ increases w.r.t $K = 1$ exhibits a diminishing trend as K increases. For example, for $N = 60$, the maximum node-sector power for $K = 2$ is approximately 2.65 times the maximum node-sector power for $K = 1$. However, the maximum node-sector powers corresponding to $K = 3$ and $K = 4$ are only 3.41 and 3.77 times the maximum node-sector power for $K = 1$.

8.3 Edge density vs. K

In this section, we provide some results on the edge densities of the power optimized topologies. For any N and a given P^{max}, let G denote the reachability graph corresponding to a certain network instance. Also, let T be the power optimized topology for that network instance, for a specific K. Then, the edge density of T is defined as the ratio of the number of bidirected edges in T to the number of bidirected edges in G. For each N, the average number of bidirected edges in the reachability graphs is shown in Table 1.

Table 5 provides a comparison of edge densities for different K and N, averaged over 100 trials. The numbers in boldface in the third column represent the ratios of the edge densities for $K = 2$ to $K = 1$. Similarly, the boldfaced numbers in the fourth and fifth columns represent the ratios of the edge densities for $K = 3$ to $K = 1$ and $K = 4$ to $K = 1$. As with the total power cost (Table 3b) or the maximum power cost (Table 4b), it can be seen that, for any N, the rate at which the edge density for $K \geq 2$ increases w.r.t $K = 1$ decreases as K increases.

Table 5: Comparison of edge densities for different K and N, averaged over 100 trials. The numbers in **boldface** in the third column represent the ratios of the edge densities for $K = 2$ to $K = 1$. Similarly, the boldfaced numbers in the fourth and fifth columns represent the ratios of the edge densities for $K = 3$ to $K = 1$ and $K = 4$ to $K = 1$. For example, for $N = 20$, the ratio of the edge densities for $K = 2$ to $K = 1$ is equal to $0.47/0.14 = 3.26$.

N	$K = 1$	$K = 2$	$K = 3$	$K = 4$
20	0.14	0.47 (**3.26**)	0.61 (**4.36**)	0.72 (**5.14**)
40	0.09	0.48 (**5.33**)	0.66 (**7.33**)	0.76 (**8.44**)
60	0.08	0.56 (**7.00**)	0.71 (**8.88**)	0.82 (**10.25**)
80	0.07	0.60 (**8.57**)	0.74 (**10.57**)	0.82 (**11.71**)
100	0.06	0.63 (**10.50**)	0.78 (**13.00**)	0.85 (**14.17**)

9 Conclusion

In this paper, we have considered the problem of K-node connected minimum power bidirectional topology optimization in wireless networks with sectored antennas. We proposed sub-optimal heuristic procedures for constructing and improving power efficient K-node connected topologies. The construction phase is based on Kruskal's algorithm for the minimum spanning tree problem. The topology improvement phase is used to remove non-essential edges from the construction phase, without affecting the desired connectivity.

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